

Pencils and nets on curves arising from rank 1 sheaves on K3 surfaces

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Abstract

Let S be a K3 surface, C a smooth curve on S with $\mathcal{O}_S(C)$ ample, and A a base-point free g_d^2 on C of small degree. We use Lazarsfeld–Mukai bundles to prove that A is cut out by the global sections of a rank 1 torsion-free sheaf \mathcal{G} on S . Furthermore, we show that $c_1(\mathcal{G})$ with one exception is adapted to $\mathcal{O}_S(C)$ and satisfies $\text{Cliff}(c_1(\mathcal{G})|_C) \leq \text{Cliff}(A)$, thereby confirming a conjecture posed by Donagi and Morrison. We also show that the same methods can be used to give a simple proof of the conjecture in the g_d^1 case.

In the final section, we give an example of the mentioned exception where $h^0(C, c_1(\mathcal{G})|_C)$ is dependent on the curve C in its linear system, thereby failing to be adapted to $\mathcal{O}_S(C)$.

1 Introduction

In the past 30 years, one central problem in the study of the existence of g_d^r 's on smooth curves has been to find connections between sheaves on K3 surfaces S and linear systems on curves C lying on S . This started with Lazarsfeld ([9]) and Tyurin ([12]) independently introducing vector-bundles $\mathcal{E}_{C,A}$ on S , depending on a smooth curve C and base-point free complete linear system A on C , providing much information on the geometry of the curve C and existence of other linear systems on the curve.

These vector-bundle techniques have given grounds for many results that have emerged lately. Among these, Knutsen has proved that both gonality and Clifford index are constant for all smooth curves in a linear system on a K3 surface, with only one particular exception for the gonality case, and that there exist only two examples of exceptional curves ([7], see also [3]); and Aprodu and Farkas proved that the Green conjecture is satisfied for all smooth curves on K3 surfaces, and at the same time found the exact dimension of g_d^1 's for the general curves in a linear system ([1]).

Lelli-Chiesa ([11]) proved a conjecture posed by Donagi and Morrison ([4]), in the case of K3 surfaces without (-2) curves, $d \leq g - 1$ and $\text{Cliff}(A) = \text{Cliff}(C)$. The conjecture is stated as follows:

Conjecture 1.1 (Donagi–Morrison, [4]). Suppose C is a smooth curve on a K3 surface S , and let A be a base-point free complete g_d^r on C such that $\rho(g, r, d) < 0$. Then there exists a line bundle D on S , adapted to $\mathcal{O}_S(C)$, such that $A \leq D|_C$ and $\text{Cliff}(D|_C) \leq \text{Cliff}(A)$.

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Here, the *Clifford index* of a line bundle A on a smooth curve C is defined as $\text{Cliff}(A) := \deg(A) - 2(h^0(C, A) - 1)$. We also mention that $\text{Cliff}(C) := \min\{\text{Cliff}(A) \mid h^0(C, A), h^1(C, A) \geq 2\}$ (but where $\text{Cliff}(C)$ is defined to be 0 for hyperelliptic curves of genus 2 or 3, and 1 for trigonal curves of genus 3). The value $\rho(g, r, d)$ is the *Brill–Noether number* and is defined as $\rho(g, r, d) := g - (r - 1)(g - d + r)$. A line bundle D on S is said to be *adapted to the line bundle L* if:

- (i) $h^0(S, D) \geq 2$ and $h^0(S, L \otimes D^\vee) \geq 2$, and
- (ii) $h^0(C, D|_C)$ is independent of the curve $C \in |L|_s$;

where $|L|_s$ denotes smooth curves in $|L|$.

The conjecture was proved in [4] for the case of g_d^1 's, and basically involved proving that c_1 of the cokernel of the maximal destabilising sequence of $\mathcal{E}_{C,A}$ satisfies the conditions of the line bundle D in the conjecture, with the exception of one special case.

Part of the conjecture was also proved by Lelli-Chiesa in [10] for the case of g_d^2 's on curves on maximal gonality and Clifford dimension 1. There, the idea was to prove that the kernel of the maximal destabilising sequence of $\mathcal{E}_{C,A}$ can be assumed to be of rank 1, and that the determinant of the cokernel is the desired line bundle D of the conjecture.

In the proof of our result, we use similar ideas. The main result states that the divisors in base-point free complete g_d^2 's for small d are equal to global sections of torsion-free sheaves of rank 1 on S restricted to C . The torsion-free sheaves arise naturally from a maximal destabilising sequence of $\mathcal{F}_{C,A} = \mathcal{E}_{C,A}^\vee$, and c_1 of these sheaves satisfy the conditions on D of the conjecture, similar to what is done in Donagi–Morrison's and Lelli-Chiesa's proofs.

Our main result is the following:

Theorem 1.2. *Let S be any K3 surface, and let L be an ample line-bundle on S . If $C \in |L|$ is smooth and A is a base-point free complete g_d^2 on C satisfying $d \leq \frac{1}{6}L^2$, then the following is satisfied:*

- (a) *There exists a linear system $|D|$ on S and a finite subscheme $\xi \subset S$ such that every divisor in $|A|$ is equal to an element in $|D \otimes \mathcal{I}_\xi|$ restricted to C , where \mathcal{I}_ξ is the ideal sheaf of ξ .*

Suppose furthermore that there do not exist an elliptic pencil E and a (-2) curve Γ satisfying both conditions $A = E|_C^{\otimes 2}$ and $(L \otimes E^{\otimes (-2)}) \cdot \Gamma = -2$. Then:

- (b) *The line bundle D found in (a) is adapted to L ; and*
- (c) $\text{Cliff}(D|_C) \leq \text{Cliff}(A)$.

Remark 1.3. In the case where there exist an elliptic pencil E and a (-2) curve Γ satisfying $A = E|_C^{\otimes 2}$ and $(L \otimes E^{\otimes (-2)}) \cdot \Gamma = -2$, we can construct examples where $E|_C^{\otimes 2}$ is dependent on the curve C in $|L|$. See Example 3.1.

The tools used in the proof of Theorem 1.2, can also be used to give a simple proof of Donagi and Morrison's result for the g_d^1 case ([4, Theorem 5.1']). Here, we avoid the special case that was considered in the original proof. Furthermore, as in Theorem 1.2, we also here prove that all divisors in $|A|$ are equal to the restriction to C of the global sections of a rank-1 torsion-free sheaf on S .

Theorem 1.4. *Let $|L|$ be any base-point free linear system on a K3 surface S . If $C \in S$ is smooth and A is a base-point free complete g_d^1 on C satisfying $\rho(g, 1, d) < 0$, then the following is satisfied:*

- (a) *There exists a linear system $|D|$ on S and a finite subscheme $\xi \subset S$ such that every divisor in $|A|$ is equal to an element in $|D \otimes \mathcal{I}_\xi|$ restricted to C , where \mathcal{I}_ξ is the ideal sheaf of ξ ;*
- (b) *the line bundle D found in (a) is adapted to L ; and*
- (c) $\text{Cliff}(D|_C) \leq \text{Cliff}(A)$.

We will be working in characteristic 0 throughout this paper.

2 Proof of theorem

The Lazarsfeld–Mukai vector bundles are defined as follows: Given a smooth curve C of genus g on S and a base-point free, complete g_d^r A on C , the vector-bundle $\mathcal{F}_{C,A}$ on S is defined as the kernel of the evaluation morphism $H^0(C, A) \otimes \mathcal{O}_S \rightarrow A \rightarrow 0$. The bundle has the following properties:

- $\text{rk}(\mathcal{F}_{C,A}) = r + 1$.
- $\det(\mathcal{F}_{C,A}) = \mathcal{O}_S(-C)$.
- $c_2(\mathcal{F}_{C,A}) = d$.
- $h^0(S, \mathcal{F}_{C,A}) = h^1(S, \mathcal{F}_{C,A}) = 0$.
- $\chi(S, \mathcal{F}_{C,A}) = 2 - 2\rho(g, r, d)$ where $\rho(g, r, d) = g - (r + 1)(g - d + r)$ is the Brill–Noether number.
- The dual, $\mathcal{F}_{C,A}^\vee$, is globally generated away from a finite set.

Note that if $\rho(g, r, d) < 0$, then $2h^0(S, \mathcal{F}_{C,A} \otimes \mathcal{F}_{C,A}^\vee) \geq \chi(S, \mathcal{F}_{C,A}) \geq 4$, and so $\mathcal{F}_{C,A}$ is then non-simple, and hence non-stable.

We will for the remainder of this paper – except in the proof of Theorem 1.4 – assume that A is a base-point free, complete g_d^2 on a smooth curve $C \in |L|$ satisfying $d \leq \frac{1}{6}L^2$. By [6, Theorem 3.4.1], $\mathcal{F}_{C,A}$ is then unstable, and there thus exists a maximal destabilising sequence

$$0 \rightarrow M \rightarrow \mathcal{F}_{C,A} \rightarrow N \rightarrow 0 \quad (1)$$

such that M is locally free, N is torsion-free and μ_L -semistable, and $M.L > -\text{rk}(M)\frac{1}{3}L^2$.

In the statements that follow, we will also be needing the dualisation of this sequence, which is

$$0 \rightarrow N^\vee \rightarrow \mathcal{F}_{C,A}^\vee \rightarrow \tilde{M} \rightarrow 0, \quad (2)$$

where \tilde{M} is torsion-free and satisfies $\tilde{M}^\vee = M$. Since $\mathcal{F}_{C,A}^\vee$ is globally generated away from a finite set, the same applies for \tilde{M} . This sequence is maximal destabilising for $\mathcal{F}_{C,A}^\vee$, and so \tilde{M} must be μ_L -semistable.

The following lemma is needed for the proof of Proposition 2.2, which (among other things) states that we can assume the rank of M to be 2. This is the most important step for the proof of Theorem 1.2.

Lemma 2.1. *Let A , C and L be as above, and consider the maximal destabilising sequence (1) of $\mathcal{F}_{C,A}$. If $\mathrm{rk}(M) = 1$, then $M.c_1(N) \geq 0$.*

Proof. Suppose $M.c_1(N) = M^\vee.c_1(N^\vee) < 0$.

We dualise the sequence (1), yielding

$$0 \rightarrow N^\vee \rightarrow \mathcal{F}_{C,A}^\vee \rightarrow M^\vee \otimes \mathcal{I}_\eta \rightarrow 0,$$

where \mathcal{I}_η is the ideal sheaf of a 0-dimensional subscheme η . Since $\mathcal{F}_{C,A}^\vee$ is globally generated away from a finite set, then so is M^\vee , and it follows that a sufficient condition for $M.c_1(N) = M^\vee.c_1(N^\vee)$ to be ≥ 0 is that $h^0(S, c_1(N^\vee)) \geq 1$.

Now, since $M.L > -\frac{1}{3}L^2$, then $c_1(N).L < -\frac{2}{3}L^2 < 0$, and so it suffices to show that $c_1(N)^2 \geq 0$, since it then follows that either $c_1(N)$ or $c_1(N^\vee)$ must be effective, and we see that it must be $c_1(N^\vee)$.

To show that $c_1(N)^2 \geq 0$, we first consider the maximal destabilising sequence (1), where $M.L > -\frac{1}{3}L^2$. Using that $M \otimes c_1(N) \cong L^\vee$, this gives us $c_1(N).L = -L^2 - M.L < -L^2 + \frac{1}{3}L^2 = -\frac{2}{3}L^2 = -\frac{2}{3}(2g - 2)$, i.e., $c_1(N^\vee)^2 + c_1(N^\vee).M^\vee = c_1(N^\vee).L > \frac{2}{3}(2g - 2)$. Since we are assuming that $c_1(N^\vee).M^\vee < 0$, it follows that $c_1(N^\vee)^2 > 0$.

The result follows. \square

Proposition 2.2. *Let A , C and L be as above, and let (1) be a maximal destabilising sequence of $\mathcal{F}_{C,A}$. Then $\mathrm{rk}(M) = 2$; $c_1(M)^2 \geq 0$; $c_2(\tilde{M}) \geq 0$, where \tilde{M} is as in (2); and $c_2(M) \geq 0$.*

Proof. Suppose $\mathrm{rk}(M) = 1$. Then N is semistable of rank 2, and so by [6, Theorem 3.4.1], $c_2(N) \geq \frac{1}{4}c_1(N)^2$. Furthermore, $M.L > -\frac{1}{3}L^2$, and so $c_1(N).L < -\frac{2}{3}L^2$. From (1), we hence get $c_2(\mathcal{F}_{C,A}) \geq M.c_1(N) + \frac{1}{4}c_1(N)^2 = \frac{1}{4}c_1(N)(c_1(N) + 4M) = \frac{1}{4}c_1(N)(-L + 3M) > \frac{2}{12}L^2 + \frac{3}{4}c_1(N).M$, and by Lemma 2.1, this is $\geq \frac{1}{6}L^2$. Since $c_2(\mathcal{F}_{C,A}) = \deg(A)$, which was assumed to be $\leq \frac{1}{6}L^2$, this gives the desired contradiction.

To prove the first two inequalities of the statement, we consider (2) and note that since \tilde{M} is globally generated away from a finite set, then the same must apply for $c_1(\tilde{M}) = c_1(M^\vee)$, and so $c_1(M)^2 = c_1(M^\vee)^2 \geq 0$. By [6, Theorem 3.4.1], we must have $c_2(\tilde{M}) \geq 0$ as a consequence.

The last statement follows by noting that $c_2(M)$ can only be negative if any exact sequence

$$0 \rightarrow D_1 \rightarrow M \rightarrow D_2 \otimes \mathcal{I}_\eta \rightarrow 0,$$

where \mathcal{I}_η is the ideal sheaf of a (possibly empty) finite subscheme, and where D_i are line-bundles, satisfies $D_1.D_2 < 0$. However, since M^\vee is globally generated away from a finite set, we can inject an effective line-bundle D'_2 into M^\vee , assume that the injection is saturated, and get

$$0 \rightarrow D'_2 \rightarrow M^\vee \rightarrow D'_1 \otimes \mathcal{I}_\zeta \rightarrow 0,$$

where ζ is a possibly empty zero-dimensional subscheme and D'_1 a line-bundle. Since M^\vee is globally generated away from a finite set, then D'_1 is also globally generated (actually everywhere since it is base-component free). But then, $D'_1.D'_2 \geq 0$, and if we dualise this sequence, we can put $(D'_i)^\vee = D_i$ and get an extension where $D_1.D_2 \geq 0$. So $c_2(M) \geq 0$. \square

In the proof of part (c) of Theorem 1.2, we will be needing the following result:

Proposition 2.3. *Suppose D_1 and D_2 are two divisors on a K3 surface, and suppose $D_2^2 > 0$. Then $D_1^2 D_2^2 \leq (D_1.D_2)^2$, with equality if and only if $(D_1.D_2)D_1 \sim D_1^2 D_2$.*

Proof. This follows from the Hodge Index Theorem (see e.g. [2, Corollary 2.16]) and [5, Chapter 1, Exercise 10], and using that numeric and linear equivalence are the same for divisors on a K3 surface. \square

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. We begin by proving part (a) of the theorem.

Let A , C and L be as in the theorem, and let $\mathcal{F}_{C,A}$ be the associated Lazarsfeld–Mukai bundle. Since $\deg(A) \leq \frac{1}{6}L^2$, it follows from [6, Theorem 3.4.1] that $\mathcal{F}_{C,A}$ is unstable, and so we obtain a maximal destabilising sequence (1).

The injection $M \hookrightarrow \mathcal{F}_{C,A}$ can be composed with $\mathcal{F}_{C,A} \hookrightarrow \mathcal{O}_S^{\oplus 3}$, yielding the following diagram, where \mathcal{G} is the cokernel:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathcal{O}_S^{\oplus 3} & \xrightarrow{\tilde{\text{ev}}} & \mathcal{G} \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \phi \\ 0 & \longrightarrow & \mathcal{F}_{C,A} & \longrightarrow & \mathcal{O}_S^{\oplus 3} & \xrightarrow{\text{ev}} & A \longrightarrow 0. \end{array} \quad (3)$$

By the snake lemma, $\ker(\phi) \cong N$, and since any torsion element of \mathcal{G} must map to 0 in A and N is torsion-free, it follows that \mathcal{G} is torsion-free. Since $\text{rk}(\mathcal{G}) = 1$, it follows that $\mathcal{G} = D \otimes \mathcal{I}_\xi$, where $D = c_1(M^\vee)$ and \mathcal{I}_ξ is the ideal sheaf of a possibly empty finite subscheme ξ .

Since ϕ is injective on global sections and $h^0(S, \mathcal{G}) \geq 3 = h^0(C, A)$, it is clear that $h^0(S, \mathcal{G}) = 3$, and that each global section of A comes from a unique global section of \mathcal{G} . The map ϕ is an element of $\text{Hom}(D \otimes \mathcal{I}_\xi, A) = \text{Hom}(\mathcal{I}_\xi, A \otimes D^\vee) = H^0(S - \xi, A \otimes D^\vee)$, implying that either $A \cong D|_C \otimes \mathcal{I}_{\xi'}$, with $\xi' = \xi \cap C$, or $h^0(C, A) > h^0(C, D \otimes \mathcal{I}_{\xi'})$. However, we have $h^0(C, A) = h^0(S, D \otimes \mathcal{I}_\xi)$, and having $h^0(S, D \otimes \mathcal{I}_\xi) > h^0(C, D \otimes \mathcal{I}_{\xi'})$ would imply that $h^0(S, D \otimes \mathcal{O}_S(-C)) > 0$, which contradicts that $c_1(M).L > -\frac{2}{3}L^2$.

We conclude that $A \cong D|_C \otimes \mathcal{I}_{\xi'}$, and that every divisor in $|A|$ comes from restricting divisors in $|D|$ to ξ and restricting to C .

Proof of part (b):

We recall that D is by definition adapted to L if $h^0(S, D) \geq 2$, $h^0(S, L \otimes D^\vee) \geq 2$, and $h^0(C, D|_C)$ is independent of the curve C in $|L|_s$.

We already know that $h^0(S, D \otimes \mathcal{I}_\xi) = 3$, so the first condition is clear. To show that $h^0(S, L \otimes D^\vee) \geq 2$, note that $D \otimes L^\vee \cong c_1(N)$, which cannot be effective since $c_1(N).L < -\frac{1}{3}L^2$. It thus suffices, by Riemann–Roch, to show that $c_1(N^\vee)^2 \geq 0$. We have $c_1(N^\vee)^2 = (L - c_1(M^\vee)).c_1(N^\vee) = L.c_1(N^\vee) - c_1(M^\vee).c_1(N^\vee) > \frac{1}{3}L^2 - c_1(N^\vee).c_1(M^\vee)$. Sequence (2) gives us $\frac{1}{6}L^2 \geq c_2(\mathcal{F}_{C,A}^\vee) = c_1(M^\vee).c_1(N^\vee) + c_2(\tilde{M}) \geq c_1(M^\vee).c_1(N^\vee)$ (using Proposition 2.2 on $c_2(\tilde{M})$), and so we can conclude that $c_1(N^\vee)^2$ is nonnegative (actually, it is $\geq \frac{1}{6}L^2$).

We now show that $h^0(C, D|_C)$ is independent of the curve C in $|L|_s$. By taking cohomology of the sequence $0 \rightarrow D \otimes L^\vee \rightarrow D \rightarrow D|_C \rightarrow 0$, we see that it suffices to show that $h^1(S, D) = 0$. This fails to happen if and only if there exists a -2 -curve Γ such that $\Gamma.D < 0$ or $D = \mathcal{O}_S(nE)$ for some positive integer n and where E is an elliptic curve (see [8, Theorem]). Since \tilde{M} is globally generated away from a finite set, then so is D (it is actually base-point free, since it is on a K3 surface and has no base-components), and so no -2 -curve can intersect D negatively. In order to prove that $h^1(S, D) = 0$, it therefore suffices to prove that $D^2 > 0$.

To prove that $D^2 > 0$, the top row of (3) shows that $0 = c_1(M).c_1(\mathcal{G}) + c_2(M) + c_2(\mathcal{G}) = -c_1(M)^2 + c_2(M) + c_2(\mathcal{G})$. Since $c_2(M) \geq 0$ by Proposition 2.2, then $c_1(M)^2 \geq c_2(\mathcal{G})$. We obviously have $c_2(\mathcal{G}) \geq 0$, and we have $c_1(M)^2 = 0$ only if $c_2(\mathcal{G}) = 0$, and hence only if $\mathcal{G} = D$, with $h^0(S, D) = 3$. However, in this case, $A = D|_C$, and we clearly see that $D = E^{\otimes 2}$ where E is an elliptic pencil.

We prove that $h^0(C', E|_{C'}^{\otimes 2}) = 3$ for all $C' \in |L|_s$ under the conditions of the theorem, given that there exists a curve $C \in |L|_s$ where $h^0(C, E|_C^{\otimes 2}) = 3$. Consider the exact sequence

$$0 \rightarrow E^{\otimes 2} \otimes L^\vee \rightarrow E^{\otimes 2} \rightarrow E|_C^{\otimes 2} \rightarrow 0.$$

If we take cohomology, we see that since $h^1(S, E^{\otimes 2}) = 1$, $h^0(S, E^{\otimes 2}) = 3$, and $h^0(C, E|_C^{\otimes 2}) = 3$, we must have $h^1(S, E^{\otimes 2} \otimes L^\vee) \leq 1$. If $h^1(S, E^{\otimes 2} \otimes L^\vee) = 0$, then $h^0(C', E|_{C'}^{\otimes 2}) = 3$ for all $C' \in |L|_s$, and we are done. If $h^1(S, E^{\otimes 2} \otimes L^\vee) = 1$, then so is $h^1(S, L \otimes E^{\otimes(-2)})$, and by [8, Theorem], we either have $(L \otimes E^{\otimes(-2)})^2 = 0$ or there exists a (-2) -curve Γ such that $(L \otimes E^{\otimes(-2)}) \cdot \Gamma \leq -2$. By assumption, $(L \otimes E^{\otimes(-2)})^2 \geq \frac{2}{3}L^2 > 0$, and so we are in the case where there exists a (-2) -curve Γ such that $(L \otimes E^{\otimes(-2)}) \cdot \Gamma \leq -2$.

We prove that $(L \otimes E^{\otimes(-2)}) \cdot \Gamma = -2$, thus contradicting the conditions of the theorem. If $(L \otimes E^{\otimes(-2)}) \cdot \Gamma < -2$, then $\Gamma^{\otimes 2}$ must be a base component of $L \otimes E^{\otimes(-2)}$. However, since $h^0(S, L \otimes E^{\otimes(-2)}) = h^0(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^{\otimes(-2)})$, then $\chi(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^{\otimes(-2)}) - \chi(S, L \otimes E^{\otimes(-2)}) \leq h^1(S, L \otimes E^{\otimes(-2)}) = 1$ (noting that no h^2 terms are positive because of the condition $E^{\otimes 2} \cdot L \leq \frac{1}{6}L^2$); and Riemann–Roch gives us $-2(L \otimes E^{\otimes(-2)}) \cdot \Gamma - 4 \leq 1$, which is impossible.

Proof of part (c):

We have $\text{Cliff}(A) = d - 4 = c_2(\mathcal{F}_{C,A}) - 4$. We must prove that $\text{Cliff}(D|_C)$ is at most equal to this.

By definition, $\text{Cliff}(D|_C) = D \cdot L - 2(h^0(C, D|_C) - 1)$. Since $D = c_1(M^\vee)$, $D \cdot L = c_1(M^\vee) \cdot c_1(N^\vee) + c_1(M^\vee)^2$ and $h^0(C, D|_C) \geq h^0(S, D) \geq \frac{1}{2}c_1(M^\vee)^2 + 2$, this immediately gives us

$$\text{Cliff}(D|_C) \leq c_1(M^\vee) \cdot N^\vee - 2 = c_2(\mathcal{F}_{C,A}) - c_2(\tilde{M}) - 2, \quad (4)$$

where we recall that \tilde{M} is as given in (2). We must prove that $c_2(\tilde{M}) \geq 2$.

Since \tilde{M} is semistable, it follows from [6, Theorem 3.4.1] that $c_2(\tilde{M}) \geq \frac{1}{4}c_1(\tilde{M})^2$. In part (b) of the proof, we showed that $c_1(\tilde{M})^2 > 0$ (since $D = c_1(\tilde{M})$). And so $c_2(\tilde{M}) \geq 1$. In the following, we will suppose $c_2(\tilde{M}) = 1$ (and hence $c_1(\tilde{M})^2 \leq 4$) and show that this yields a contradiction.

First note that, by taking cohomology of (2) and recalling that $h^1(S, \mathcal{E}_{C,A}) = h^2(S, \mathcal{E}_{C,A}) = 0$, we see that $h^1(S, \tilde{M}) = h^2(S, \tilde{M}) = 0$. Also, since \tilde{M} is of rank 2 and globally generated away from a finite set, it must sit inside an exact sequence

$$0 \rightarrow R_1 \otimes \mathcal{I}_\nu \rightarrow \tilde{M} \rightarrow R_2 \otimes \mathcal{I}_\eta \rightarrow 0, \quad (5)$$

where R_i are line-bundles and ν and η finite subschemes. We can furthermore assume that R_1 is effective since \tilde{M} has global sections, and R_2 is globally generated and $R_2 \otimes \mathcal{I}_\eta$ globally generated away from a finite set; and hence $R_1 \cdot R_2 \geq 0$ and $\text{length}(\nu), \text{length}(\eta) \leq 1$. Note that $R_1 \cdot R_2 = 1 - \text{length}(\eta) - \text{length}(\nu)$. Also, [8, Theorem] gives us that $h^1(S, c_1(\tilde{M})) = 0$.

Case: $\text{length}(\eta) = 1$. In this case, $R_1.R_2 = \text{length}(\nu) = 0$, and since at least one R_i must satisfy $R_i^2 > 0$, Proposition 2.3 yields that $R_1^2 R_2^2 \leq 0$, and so either $R_1^2 < 0$, or $R_1^2 = 0$. (If $R_1^2 > 0$ with $R_2^2 = 0$, we get $R_2 = \mathcal{O}_S$, and then $R_2 \otimes \mathcal{I}_\eta$ has no global sections.) If $R_1^2 < 0$, then also $R_1.c_1(\tilde{M}) = R_1.(R_1 \otimes R_2) < 0$, and so R_1 is a base component of $c_1(\tilde{M})$. However, since \tilde{M} is globally generated away from a finite set, then so must $c_1(\tilde{M})$, and we get a contradiction.

If $R_1^2 = 0$, then $(R_1 \otimes R_2)^2 = R_2^2$, and putting $\mathcal{O}_S(D_1) = R_2$ and $\mathcal{O}_S(D_2) = R_1 \otimes R_2$, we get equality in Proposition 2.3, and so $R_1 = \mathcal{O}_S$ (since numerical and linear equivalence is the same for line bundles on K3 surfaces). However, in that case, taking cohomology of (5) gives us that $h^1(S, R_2 \otimes \mathcal{I}_\eta) = 1$, while cohomology of the sequence

$$0 \rightarrow R_2 \otimes \mathcal{I}_\eta \rightarrow R_2 \rightarrow \mathcal{O}_\eta \rightarrow 0$$

yields $h^1(S, R_2 \otimes \mathcal{I}_\eta) \geq \text{length}(\eta) - h^0(S, R_2) + h^0(S, R_2 \otimes \mathcal{I}_\eta) \geq \text{length}(\eta) - 1$, so that $\text{length}(\eta) \geq 2$, a contradiction.

Case: $\text{length}(\nu) = 1$. This case is similar to the previous case. Here we also have $R_1.R_2 = 0$, and in addition, $\text{length}(\eta) = 0$. Here, we cannot have $R_1^2 > 0$ with $R_2^2 = 0$, because we get $R_2 = \mathcal{O}_S$, and dualising (5) would imply that $\tilde{M}^\vee = M$ has global sections, a contradiction. So the two alternatives are $R_1^2 < 0$ or $R_1^2 = 0$, as in the previous case. We cannot have $R_1^2 < 0$ for the same reason as in the previous case. If $R_1^2 = 0$ with $R_2^2 > 0$, we get $R_1 = \mathcal{O}_S$ as in the previous case, and $h^1(S, R_2) = 1$. However, since R_2 is globally generated with positive self-intersection, this is impossible by [8, Theorem].

Case: $\text{length}(\eta) = \text{length}(\eta) = 0$. In this case, $R_1.R_2 = 1$, and since self intersection on a K3 surface is always even, Proposition 2.3 yields that $R_1^2 R_2^2 \leq 0$. If $R_1^2 < 0$, then it must be ≤ -2 , and we get $R_1.(R_1 \otimes R_2) \leq -1$, and so R_1 is a base component of $c_1(\tilde{M})$, which contradicts $c_1(\tilde{M})$ being globally generated. It follows that $R_1^2 \geq 0$.

Having $R_1^2, R_2^2 \geq 0$ implies that $(R_2 \otimes R_1^\vee)^2 \geq -2$, and it follows from Riemann–Roch that either $R_2 \geq R_1$ or $R_1 \geq R_2$. The semistability of \tilde{M} implies that $R_2 \geq R_1$. Since R_2 is globally generated, $(R_2 - R_1).R_2 \geq 0$, and so $R_2^2 \geq 2$ and $c_1(\tilde{M})^2 \geq 4$. Since we originally had $c_1(\tilde{M})^2 \leq 4$ (as a consequence of assuming $c_2(\tilde{M}) = 1$), equality follows, together with $R_2^2 = 2$ and $R_1^2 = 0$. By [8, Theorem], $h^1(S, R_2) = 0$, and since $h^2(S, \tilde{M}) = 0$, we get $h^2(S, R_1) = 0$, and so $R_1 = E^{\otimes n}$ where E is an elliptic pencil. Since $R_1.R_2 = 1$, then $n = 1$.

Note that since $R_2 \otimes R_1^\vee > 0$ and \tilde{M} is semistable, then \tilde{M} must be a proper extension of R_1 and R_2 . The dimension of isomorphism classes of non-trivial extensions is $\text{Ext}_{\mathcal{O}_S}^1(R_2, R_1) - 1 = h^1(S, R_2 \otimes R_1^\vee) - 1$, and so $h^1(S, R_2 \otimes R_1^\vee) > 0$. By [8, Theorem], this implies that either $R_2 \cong E \otimes E'$ where E' is an elliptic pencil satisfying $E'.E = 1$; or $(R_2 \otimes R_1^\vee).\Gamma \leq -2$ for some (-2) -curve Γ , implying that $R_2 = E \otimes B \otimes \Gamma^{\otimes m}$, where $m \geq 1$ is an integer and $B \geq 0$ is a possibly trivial line bundle satisfying $B.\Gamma \geq 0$.

If $R_2 = E \otimes E'$, then $h^1(S, R_2 \otimes R_1^\vee) = 0$, a contradiction.

If $R_2 = E \otimes B \otimes \Gamma^{\otimes m}$, note that since $R_1 = E$, then $1 = R_1.R_2 = E.B + mE.\Gamma \geq mE.\Gamma$. Since $(R_2 \otimes R_1).\Gamma \geq 0$ (recall that $R_1 \otimes R_2$ is globally generated) and $(R_2 \otimes R_1^\vee).\Gamma \leq -2$, we get $R_1.\Gamma \geq 1$. However, since $R_2.\Gamma \geq 0$ (recall that R_2 is globally generated), this means that $(R_2 \otimes R_1).\Gamma \geq 1$ instead of ≥ 0 , and we end up with $R_1.\Gamma \geq 2$. But then $R_2.R_1 \geq mE.\Gamma \geq 2$, a contradiction.

We conclude that $\text{Cliff}(D|_C) \leq \text{Cliff}(A)$. □

Proof of Theorem 1.4. The proof of Theorem 1.4 uses exactly the same techniques as in the proof of Theorem 1.2. We include it here for the sake of completion.

The condition on C and A are that $\rho(g, 1, d) < 0$. In this case, it follows that $\mathcal{F}_{C,A}$ is non-simple, and hence non-stable.

Part (a) is proved using the same diagram as in (3), the only difference being that $\text{rk}(M) = 1$, and that $M.L \geq -\frac{1}{2}L^2$ and $c_1(N).L \leq -\frac{1}{2}L^2$. The latter inequality implies that N has no global sections, and so ϕ is injective on global sections. It follows, from the arguments in the proof of Theorem 1.2, that each global section of A comes from a unique global section of \mathcal{G} , and that the map must be the restriction map to C .

We now prove part (b): Following the proof of Theorem 1.2 (b), it is clear that $h^0(S, D) \geq 2$. We have $h^1(S, D) = h^0(S, L \otimes D^\vee) = h^0(S, c_1(N)^\vee)$. To prove that the latter is ≥ 2 , it suffices by Riemann–Roch to show that $c_1(N)^2 \geq 0$. We have $c_1(N).(M \otimes c_1(N)) = c_1(N).L^\vee \geq \frac{1}{2}L^2 = \frac{1}{2}M^2 + M.c_1(N) + \frac{1}{2}c_1(N)^2$, and so $\frac{1}{2}c_1(N)^2 \geq \frac{1}{2}M^2$. Since $c_1(\tilde{M}) = M^\vee$ and \tilde{M} is globally generated away from a finite set, then M^\vee is globally generated, and so $M^2 = (M^\vee)^2 \geq 0$, and we can conclude that $h^1(S, D) \geq 2$.

The argument that $h^0(C, D|_C)$ is independent of the curve C in $|L|_s$ is similar to the argument in the proof of Theorem 1.2. We see that no (-2) -curve can intersect D negatively, and so $h^1(S, D)$ can be positive only if $D^2 = 0$. We see that $0 = M.c_1(\mathcal{G}) + c_2(\mathcal{G})$, and so $D^2 = M^2 \geq c_2(\mathcal{G})$. Thus, $D^2 = 0$ if and only if $\mathcal{G} = D$. As a consequence, $h^0(S, D) = 2$, and so we must have $D = E$ where E is an elliptic pencil. In that case, $h^1(S, D) = 0$, and we conclude that $h^0(C, D|_C)$ is independent of the curve C in $|L|_s$.

To prove (c), we have $\text{Cliff}(A) = d - 2 = c_2(\mathcal{F}_{C,A}) - 2$; and $\text{Cliff}(D|_C) = D.L - 2(h^0(C, D|_C) - 1) \leq c_2(\mathcal{F}_{C,A}) - c_2(\tilde{M}) - 2 \leq c_2(\mathcal{F}_{C,A}) - 2$, and so $\text{Cliff}(D|_C) \leq \text{Cliff}(A)$, as desired. \square

3 An example of a linear system $|L|$ where $h^0(C, E|_C^{\otimes 2})$ depends on the curve C in $|L|_s$

In this section, we give an example of a case where $h^0(C, E|_C^{\otimes 2})$ depends on the curve C in $|L|_s$.

Example 3.1. Suppose we have an elliptic pencil E and a (-2) -curve Γ satisfying $E.\Gamma = 2$. This intersection can occur e.g. when S is the intersection of a quadric Q_2 and cubic Q_3 in \mathbb{P}^4 , such that for some hyperplane section H we get $Q_2 \cap H = P_1 + P_2$ where P_i are planes, $Q_3 \cap P_1$ consists of a line ℓ and conic Γ , and $Q_3 \cap P_2$ consists of an elliptic curve E . In this situation, Q_3 intersects $P_1 \cap P_2$ in three points, ℓ intersects $P_1 \cap P_2$ in one point, Γ intersects $P_1 \cap P_2$ in two points, and hence, $\Gamma.E = 2$.

Now let $L = aE + (a-1)\Gamma$ where $a \geq 3$ is an integer. (We let $a \geq 7$ if we wish the condition $\deg(A) \leq \frac{1}{6}L^2$ from Theorem 1.2 to be satisfied, but the example works for all $a \geq 3$.) It is clear that $|L|$ contains (smooth) irreducible curves by comparing $h^0(S, L)$ with $h^0(S, L \otimes \Gamma^\vee)$ and $h^0(S, L \otimes E^\vee)$, noting that h^1 of all of these line-bundles is zero, by [8, Theorem].

Consider the exact sequence

$$0 \rightarrow L^\vee \otimes E^{\otimes 2} \rightarrow E^{\otimes 2} \rightarrow E|_C^{\otimes 2} \rightarrow 0, \quad (6)$$

where C is a smooth curve in $|L|$. We argue that $h^1(S, L^\vee \otimes E^{\otimes 2}) = 1$: Note that this equals $h^1(S, L \otimes E^{\otimes (-2)})$, and by [8, Theorem], $h^1(S, L \otimes E^{\otimes (-2)}) > 0$ while $h^1(S, L \otimes E^{\otimes (-2)} \otimes \Gamma^\vee) = 0$.

By comparing $\chi(S, L \otimes E^{\otimes(-2)})$ with $\chi(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^\vee)$, and using that $h^0(S, L \otimes E^{\otimes(-2)}) = h^0(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^\vee)$, we get that $h^1(S, L \otimes E^{\otimes(-2)}) = 1$.

Now tensor (6) with Γ and take cohomology. We see that $H^0(S, E^{\otimes 2} \otimes \Gamma) \cong H^0(C, E^{\otimes 2} \otimes \Gamma|_C)$, and can therefore conclude that the linear system $|E|_C^{\otimes 2}|$ is found precisely by considering divisors in $|E^{\otimes 2} \otimes \Gamma|$ that, restricted to C , are zero in $\Gamma \cap C$.

Note that divisors in $|E^{\otimes 2} \otimes \Gamma|$ that have Γ as a component, will not cut out any extra divisors in $|E|_C^{\otimes 2}|$ apart from those already cut out by $|E^{\otimes 2}|$ on S . We must therefore consider curves in $|E^{\otimes 2} \otimes \Gamma|$ that do not have Γ as a component, but still cut through C exactly where C intersects Γ .

We now have two situations: First of all, suppose J is a curve in $|E^{\otimes 2} \otimes \Gamma|$ that does not have Γ as a component. This curve intersects Γ in two points. Since $h^0(S, L) - 2 > h^0(S, L \otimes \Gamma^\vee)$, it is easy to find an irreducible, smooth curve C' in $|L|$ that passes through $J \cap \Gamma$, and hence, $J \cap C' - \Gamma \cap C'$ is an effective divisor in $|E|_C^{\otimes 2}|$ which is not cut out by a divisor in $|E^{\otimes 2}|$. We conclude that $h^0(C', E|_{C'}^{\otimes 2}) = 4$.

It remains to prove that there exists a smooth, irreducible curve $C'' \in |L|$ where the above situation does not occur. It then suffices to find a curve C'' such that no divisor J in $|E^{\otimes 2} \otimes \Gamma|$ satisfies $J \cap \Gamma = C'' \cap \Gamma$. Consider the exact sequence

$$0 \rightarrow E^{\otimes 2} \rightarrow E^{\otimes 2} \otimes \Gamma \xrightarrow{\psi} (E^{\otimes 2} \otimes \Gamma)|_\Gamma \rightarrow 0.$$

Taking cohomology, we note that $h^1(S, E^{\otimes 2}) = 1$ while $h^1(S, E^{\otimes 2} \otimes \Gamma) = 0$, implying that ψ is not surjective on global sections. This means that there is one dimension of divisors $Z \in |(E^{\otimes 2} \otimes \Gamma)|_\Gamma|$ that are not cut out by any of the divisors in $|E^{\otimes 2} \otimes \Gamma|$. From the argument above, it follows that any curve C'' that cuts out such a divisor Z on Γ will satisfy $h^0(C'', E|_{C''}^{\otimes 2}) = 3$.

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References

- [1] M. Aprodu and G. Farkas. Green's conjecture for curves on arbitrary K3 surfaces. *Compos. Math.*, 147:839–851, 2011.
- [2] W.P. Barth, K. Hulek, C.A.M. Peters, and A. van de Ven. *Compact Complex Surfaces*, volume 4. Springer Verlag Berlin Heidelberg, 2004.
- [3] C. Ciliberto and G. Pareschi. Pencils of minimal degree on curves on a K3 surface. *J. Reine Angew. Math.*, 460:14–36, 1995.
- [4] R. Donagi and D.R. Morrison. Linear systems on K3 sections. *J. Diff. Geom.*, 29:49–64, 1989.
- [5] R. Friedman. *Algebraic Surfaces and Holomorphic Vector Bundles*. Springer Verlag, 1998.
- [6] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Springer, 2010.
- [7] A.L. Knutsen. On two conjectures for curves on K3 surfaces. *Internat. J. Math.*, 20:1547–1560, 2009.

- [8] A.L. Knutsen and A.F. Lopez. A sharp vanishing theorem for line bundles on K3 or Enriques surfaces. *Proc. Amer. Math. Soc.*, 135(11):3495–3498, 2007.
- [9] R. Lazarsfeld. Brill–Noether–Petri without degenerations. *J. Diff. Geom.*, 23:299–307, 1986.
- [10] M. Lelli-Chiesa. Stability of rank-3 Lazarsfeld–Mukai bundles on K3 surfaces. *Proc. Lond. Math. Soc. (3)*, 107(2):451–479, 2013.
- [11] M. Lelli-Chiesa. Generalized lazarsfeld–mukai bundles and a conjecture of donagi and morrison. *Adv. Math.*, 268:529–563, 2015.
- [12] A.N. Tyurin. Cycles, curves and vector bundles on an algebraic surface. *Duke Math. J.*, 54(1):1–26, 1987.

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